ON FINITENESS CONDITIONS IN THE MECHANICS OF CONTINUOUS MEDIA. STATIC PROBLEMS OF THE THEORY OF ELASTICITY

(OB USLOVIIAKH KONECHNOSTI V MEKHANIKE SPLOSHNYKH SRED. STATICHESKIE ZADACHI TEORII UPRUGOSTI)

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In different areas of the mechanics of continuous media one encounters in a number of cases the situation where the solution of the basic differential equations of the problem, which satisfies the initial and boundary conditions, is not unique but is only correct up to a constant parameter (sometimes up to several such parameters or even up to one or several functions of the independent variables of the problem). In order to determine the values of these parameters and generally undetermined elements of a given problem, one resorts to additional conditions, which sometimes are stated in the form of new independent hypotheses (postulates) on the finiteness [boundedness] of velocities, stresses, etc., supported by additional physical considerations.

One can cite many examples of problems of this kind. One of them is the problem of flow past a wing with a sharp trailing edge where one uses the Zhukovskii-Chaplygin condition on the finiteness of the velocity at the trailing edge of the wing in order to determine the unknown constant parameter, which is the circulation. In addition to that, the group of similar problems contains the contact problem of the theory of elasticity, and, in particular, the problem of the penetration of an elastic body by rigid punches [dies] where one can use the Muskhelishvili condition on the finiteness of stresses at the contour of the contact area to determine this unknown contour. We should also note the problem of the theory of cracks in brittle bodies, where one uses the Khristianovich condition on the finiteness of stresses at the crack contour in order to determine the unknown contour of the crack boundary.

One can encounter, however, cases where it is not possible to formulate a condition, which would yield a unique solution, in the form of a finiteness requirement on one or another set of quantities. In these cases the determination of a unique solution is very difficult. Such a situation arose, for instance, in the problem of the stationary displacement of one fluid by another in a porous Hele-Shaw shell, which was studied recently by Taylor and Saffman.

It appears to be possible to establish a general form of additional conditions that yield a unique solution in similar cases. These conditions, and, in particular, all above-mentioned finiteness conditions are not independent physical hypotheses. They are obtained from fundamental integral principles of mechanics just as properly as the basic differential equation and the boundary conditions. The ability to obtain additional conditions, which yield a unique solution, from integral principles of mechanics is, obviously, a fact of completely general value. This fact once again points out the advantages of formulating problems of mechanics of continuous media in terms of integral principles.

In the present paper such a study is made for static problems of the theory of elasticity*; in a subsequent paper this study will be conducted in detail for hydrodynamic problems and dynamic problems of the theory of elasticity.

1. The general form of additional conditions in static problems of the theory of elasticity. Now, let us assume that the differential equations and boundary conditions of the problem determine the *equilibrium state of the elastic system* under study in a *non-unique* fashion, i.e. the solution contains some constant parameters or functions of independent variables of the problem which remain undetermined. Let us denote the set of the undetermined elements of the solution by M; the set of the undetermined elements of the solution varied by some allowable means shall be denoted by $M + \delta M$. Let $\mathbf{u}$ represent the displacement field of the studied elastic system, which corresponds to some fixed M; $\delta_1\mathbf{u}$ is a variation of this field according to the geometric constraints imposed upon the system and corresponding to the same fixed M, and $\delta_2\mathbf{u}$ is a variation of the displacement field corresponding to the variation δM of the set of undetermined elements.

The state characterized by the displacement field $\mathbf{u} + \delta_1\mathbf{u} + \delta_2\mathbf{u}$ is a possible state of the elastic system. The principle of virtual displace-

* During a discussion of this paper at a seminar in hydromechanics at MGU, after it went to press, L.I. Sedov kindly informed me that in his course of lectures a general thermodynamic study of arbitrary models of elastic bodies considering additional physico-chemical parameters is being undertaken, where the values of these parameters are also obtained from the conditions of the extremum of the free energy or the internal energy of the system. (Note added in proof).

ments, which represents the most general formulation of the fundamental law of statics of elastic systems, can be written in the form

$$\delta W - \delta A = 0 \tag{1.1}$$

where δW is the variation of the elastic potential W of the system, and δA is the variation of the work of the external forces for a given virtual [admissible] state of the elastic system. Relation (1.1) can be written in the form

$$\delta_1 W - \delta_1 A + \delta_2 W - \delta_2 A = 0 \tag{1.2}$$

where $\delta_1 W$ and $\delta_1 A$ are, according to the previous notation, variations corresponding to the variation $\delta_1 \mathbf{u}$ of the displacement field with a fixed M, and $\delta_2 W$ and $\delta_2 A$ are variations corresponding to some variation δM. In view of the independence of the variations $\delta_1 \mathbf{u}$ and δM, the following relations follow from (1.2):

$$\delta_1 W - \delta_1 A = 0 \tag{1.3}$$

$$\delta_2 W - \delta_2 A = 0 \tag{1.4}$$

From relation (1.3) we obtain in the usual manner [1] the differential equations and boundary conditions of the problem, which correspond to the arbitrary fixed set of undetermined elements M. But if the field satisfies the differential equations of equilibrium and the boundary conditions then Clapeyron's [1] theorem holds true, according to which

$$2W = A \tag{1.5}$$

for arbitrary M, from which it follows immediately that

$$2\delta_2 W = \delta_2 A \tag{1.6}$$

Upon substitution of (1.6) into (1.4) we obtain

$$\delta_2 W = 0 \tag{1.7}$$

This relation is a general condition that determines the set of the undetermined elements of the problem. Thus, that set of undetermined elements M is concretely realized for which the elastic potential of the system takes on an extremum value. In particular, if the solution of the differential equations with the corresponding boundary conditions appears to be determined correctly up to a finite number of constants $c_1, \ldots, c_n$, then we obtain from condition (1.7)

$$\frac{\partial W}{\partial c_i} = 0 \qquad (i = 1, \ldots, n) \tag{1.8}$$

In the general case, when the state of the elastic system is also characterized by a system of functions of independent variables of the problem $f_1, \ldots, f_m$, which cannot be determined from the differential equations and boundary conditions, one has to adopt from relation (1.7) for their determination direct methods of variational calculus or integrate the variational Euler equations.

The set of undetermined elements is usually intimated by the conditions of the studied problem. However, one should note that the choice of the indicated set is substantially related to the chosen idealized scheme [model] of the phenomenon so that for an unfortunate choice of this idealized scheme, the condition of the extremum (1.7) can give no actually realizable states. Such a state of affairs is, in general, characteristic for any general theoretical approach in problems of mechanics (and not only mechanics), in particular, for instance, for dimensional analysis. It is appropriate to emphasize once more that no general theoretical approaches can do without a preceding stage of establishing an adequate model of the phenomenon to be studied.

2. The condition of the finiteness of stresses in the contact problem. Now we shall illustrate the general results obtained by some examples which are of independent interest. Let us study the problem of the penetration of an elastic half-space by a rigid punch with a curved base in the absence of friction forces. This problem was studied by Muskhelishvili (see [2], Sect. 115).

The punch, generally speaking, is non-symmetric so that the area of contact is given by the coordinates of its ends $x = a$ and $x = b > a$. The conditions of finiteness of the stresses at the edges of the contact area $x = a$ and $x = b$ are of the form [2]

$$\int_a^b \frac{f'(t)\,dt}{\sqrt{(b-t)(t-a)}} = 0, \qquad \int_a^b \frac{tf'(t)\,dt}{\sqrt{(b-t)(t-a)}} = \frac{2(1-\nu^2)P_0}{E} \tag{2.1}$$

where $y = f(x)$ is the equation of the base surface of the punch, and P_0 is the resultant force which presses the punch against the body. Let us derive conditions (1.8), which are applicable to the problem studied. The pressure under the punch, aside from the dependence upon the choice of a and b, is given by the following relation [2]:

$$p(x) = \frac{E}{2\pi(1-\nu^2)\sqrt{Q(x)}} \int_a^b \frac{\sqrt{Q(t)}\,f'(t)\,dt}{t-x} + \frac{P_0}{\pi\sqrt{Q(x)}} \tag{2.2}$$

where the integral is understood in the sense of its principal value, and

$$Q(x) = (b-x)(x-a)$$

One can show that the displacement under the punch is given by

$$v(x) = f(x) + c \tag{2.3}$$

where c is a constant determined from the relation

$$c = \frac{2(1-\nu^2)P_0}{\pi E} \ln \frac{2}{c_2} + \frac{1}{\pi} \int_0^\pi f(c_1 + c_2 \cos\lambda)\, d\lambda, \quad c_1 = \frac{1}{2}(b+a),$$
$$c_2 = \frac{1}{2}(b-a)$$

After a number of transformations the expression for the elastic potential can be written in the form

$$U = \frac{1}{2}\int_a p(x)\, v(x)\, dx = \frac{1}{2} P_0 c + \frac{P_0}{2\pi}\int_0^\pi f(c_1 + c_2\cos\lambda)\, d\lambda -$$
$$- \frac{Ec_2^2}{4\pi(1-\nu^2)} \int_0^\pi\int_0^\pi f'(c_1 + c_2\cos\lambda)\, f'(c_1 + c_2\cos\theta)\sin\lambda\sin\theta \ln \frac{\sin\frac{1}{2}(\lambda-\theta)}{\sin\frac{1}{2}(\lambda+\theta)}\, d\lambda d\theta \tag{2.4}$$

where we used

$$x = c_1 + c_2\cos\theta, \qquad t = c_1 + c_2\cos\lambda \tag{2.5}$$

Let c_1 and c_2 be the determining constant parameters so that, in view of (1.8), the following conditions can be fulfilled:

$$\frac{\partial U}{\partial c_1} = 0, \qquad \frac{\partial U}{\partial c_2} = 0 \tag{2.6}$$

Differentiation of (2.4) and integration by parts yields

$$\frac{\partial U}{\partial c_1} = \frac{Ec_2}{2\pi(1-\nu^2)}\left[\int_0^\pi f'(c_1 + c_2\cos\lambda)\, d\lambda\right]\left[\int_0^\pi f'(c_1 + c_2\cos\lambda)\cos\lambda\, d\lambda\right] \tag{2.7}$$

$$\frac{\partial U}{\partial c_2} = -\frac{Ec_2}{4\pi(1-\nu^2)}\left\{\left[\int_0^\pi f'(c_1 + c_2\cos\lambda)\, d\lambda\right]^2 + \left[\int_0^\pi f'(c_1 + c_2\cos\lambda)\cos\lambda d\lambda\right]^2\right\} +$$
$$+ \frac{P_0}{\pi}\int_0^\pi f'(c_1 + c_2\cos\lambda)\cos\lambda d\lambda - \frac{P_0^2(1-\nu^2)}{\pi c_2 E} \tag{2.8}$$

From this and from conditions (2.5) we find

$$\int_0^\pi f'(c_1 + c_2\cos\lambda)\, d\lambda = 0, \qquad \int_0^\pi f'(c_1 + c_2\cos\lambda)\cos\lambda d\lambda = \frac{2(1-\nu^2)P_0}{Ec_2} \tag{2.9}$$

Changing the variable to t by means of Formula (2.5) leads to the relations (2.1). Thus, the conditions of the finite stresses at the edges of the contact area are obtained here corresponding to the general statements developed above, originating from the principle of virtual displacements.

3. The condition of the finiteness of stresses in the crack problem. Now we shall study the problem of the straight crack in an infinite body with an arbitrary tearing force applied symmetrically to the crack line under the conditions of plane strain*. The line of the crack and its extension will be chosen as the x-axis so that the ends of the crack are at $x = a$ and $x = b$. Let the tearing forces, which we shall assume to be applied at the surface of the crack**, and the cohesion forces acting in the end-region of the crack produce at the surface of the crack normal stresses which are distributed according to the law $\sigma_y = -g(x)$, while the shear stresses at the surface of the crack are equal to zero. As we showed earlier [3], in order to satisfy the condition of finite stresses at the ends of the crack and to have a smooth joining of the opposite edges of the crack it is necessary and sufficient to satisfy the following conditions at the edges:

$$\int_a^b g(x)\sqrt{\frac{b-x}{x-a}}\,dx = 0, \int_a^b g(x)\sqrt{\frac{x-a}{b-x}}\,dx = 0 \qquad (3.1)$$

We shall show that these conditions follow from conditions (1.8). When using the results from [3] one can show that the normal displacement of the points on the surface of the crack is

$$v = \frac{(1-\nu^2)(b-a)}{\pi E}\int_0^\pi g\left[\frac{1}{2}(b+a)+\frac{1}{2}(b-a)\cos\lambda\right]\sin\lambda\ln\frac{\sin\frac{1}{2}(\lambda-\theta)}{\sin\frac{1}{2}(\lambda+\theta)}\,d\lambda \qquad (3.2)$$

where E is Young's modulus, ν is Poisson's ratio of the material, and the angle θ is given by $x = 1/2(b + a) + 1/2(b - a)\cos\theta$. The expression for the elastic potential becomes

$$W = \int_a^b g(x)\,v(x)\,dx = \frac{(1-\nu^2)(b-a)^2}{2\pi E}\int_0^\pi g\left[\frac{1}{2}(b+a) + \right.$$

$$\left. + \frac{1}{2}(b-a)\cos\theta\right]\sin\theta d\theta\int_0^\pi g\left[\frac{1}{2}(b+a) + \right.$$

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* The following discussion (Sections 3 and 4) refers, strictly speaking, to the case of reversible cracks.

** This simplification does not reduce the generality of the analysis since the case in which the tearing forces are applied inside the crack can be easily reduced to the one studied here [3].

$$+\frac{1}{2}(b-a)\cos\lambda\Big]\sin\lambda\ln\frac{\sin\frac{1}{2}(\lambda-\theta)}{\sin\frac{1}{2}(\lambda+\theta)}d\lambda \tag{3.3}$$

We shall choose the coordinates of the ends of the crack a and b as the determining constant parameters, and then, because of (1.8), the following conditions should be satisfied:

$$\frac{\partial W}{\partial a}=0, \qquad \frac{\partial W}{\partial b}=0 \tag{3.4}$$

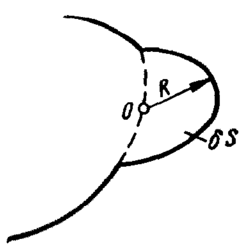
Differentiation and integration by parts, leaving out intermediate calculations, yields

$$\frac{\partial W}{\partial a}=\frac{2(1-\nu^2)(b-a)}{\pi E}\left\{\int_0^\pi g\left[\frac{1}{2}(b+a)+\frac{1}{2}(b-a)\cos\theta\right]\sin^2\frac{\theta}{2}d\theta\right\}^2=$$
$$=\frac{2(1-\nu^2)}{\pi E(b-a)}\left\{\int_0^\pi g(x)\sqrt{\frac{b-x}{x-a}}dx\right\}^2 \tag{3.5}$$

$$\frac{\partial W}{\partial b}=\frac{2(1-\nu^2)(b-a)}{\pi E}\left\{\int_0^\pi g\left[\frac{1}{2}(b+a)+\frac{1}{2}(b-a)\cos\theta\right]\cos^2\frac{\theta}{2}d\theta\right\}^2=$$
$$=\frac{2(1-\nu^2)}{\pi E(b-a)}\left\{\int_a^b g(x)\sqrt{\frac{x-a}{b-x}}dx\right\}^2 \tag{3.6}$$

From this and (3.4) we obtain the additional conditions (3.1) which determine the parameters a and b and express the conditions of finite stresses and a smooth joining of the opposite sides of the crack at its ends.

4. General proof of the condition of the finiteness of stresses. We shall present now a general proof of the condition of the finiteness of stresses. Let us consider some point O at the contour of a crack (see figure). In the neighborhood of this point the distribution of stresses and displacements can be assumed to be that of plane strain. Using the results from [3] one can easily show that, generally speaking, for an arbitrary crack contour which does not satisfy condition (1.7), the distribution of the tearing stresses σ_y in the plane of the crack near point O and the distribution of displacements v normal to this plane near point O have, respectively, the form



$$\sigma_y=N\left(\frac{1}{2\pi\sqrt{s}}\right)+O(1)+O(\sqrt{s}) \tag{4.1}$$

$$v = \frac{2(1-\nu^2)\sqrt{s}}{\pi E} N + O(s^{3/2}) \tag{4.2}$$

where s is the distance from the point in the medium studied to point O and N is some value which depends on the forces acting and the contour of the crack. In all the cases analyzed up to now, the condition of the finiteness of stresses was represented by the vanishing of the value N, which at the same time can be seen to assure a smooth joining of the opposite sides of the crack at its ends.

Let us vary now the undetermined element, the contour of the crack, such that the new contour is represented by a circular arc of some small radius R with its center at O, which lies in the plane of the crack up to the intersection with the original contour, and outside the arc it would be unchanged. Because of the smallness of the radius R, the distribution of the displacements v near every new point of the contour will be also of the form (4.2), but s will now be the distance measured from this new point. Thus, the distribution of the normal displacements at the new part of the crack will be

$$v = \frac{2(1-\nu^2)\sqrt{R-r}}{\pi E} N + O[(\sqrt{R-r})^3] \tag{4.3}$$

where r is the distance from point O to the point under consideration.

It can be easily seen that the corresponding variation of the elastic potential is equal to*

$$\delta_2 W = \iint\limits_{\delta S} \sigma_y v dS = \frac{(1-\nu^2) N^2}{\pi^2 E} \int\limits_0^R \sqrt{\frac{R-r}{r}}\, \pi r dr = \frac{(1-\nu^2) N^2}{4\pi E} \delta S \tag{4.4}$$

where δS is the variation of the crack area. Obviously, condition (1.7), i.e. $\delta_2 W = 0$, is satisfied if and only if $N = 0$. But when $N = 0$, as can be seen from Formulas (4.1) and (4.2), the finiteness of the stresses is assured at the same time as the smoothness of the joining of the opposite sides of the crack at point O. Thus, in the general case, from (1.7) follows the finiteness of stresses and the smoothness of the joining of the opposite sides of the crack at its ends.

Let us remark once more that the hypothetical form of this condition was first stated by Khristianovich. Note also that in connection with

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* The corresponding calculation for the plane case in connection with the condition of the finiteness of stresses was carried out in a paper by Irwin [4].

the mechanical statement of the problem the assertion of a wedge-shaped form of the crack near its end was made by Rebinder even earlier.

Thus, the condition of the finiteness of the stresses at the end of the crack and the smooth joining of the opposite sides of the crack at its ends was obtained from the fundamental principle of statics – the principle of virtual displacements. Because of that, the statements of problems in the theory of equilibrium cracks in brittle bodies can be limited to two basic hypotheses [3]: the hypothesis on the smallness of the end-region of the crack where forces of interaction of the opposite sides of the crack depending on the dimensions of the entire crack are acting, and the hypothesis on the autonomy of this region (i.e. the hypothesis on the independence of the distribution of the normal displacements in that region for a given material with given conditions regarding the acting forces).

The general proof given here can be applied without any essential changes to the contact problem of the theory of elasticity and its particular case, the problem of the penetration of an elastic body by punches.

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